It must be noted that, since the function U_1 is odd with respect to the first argument, the formulas (2.5) are identical with the result obtained in [2].

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Translated by L. K.

UDC 532.517.4

KÁRMÁN HYPOTHESES AND POWER LAWS FOR THE VARIATION OF ENERGY AND LINEAR SCALE OF TURBULENCE

PMM Vol. 39, № 3, 1975, pp. 567-573

A. I. KORNEEV
(Moscow)

(Received July 5, 1974)

We consider the connection between the Kármán hypotheses concerning "self-preservation" (self-similarity) of correlation functions and power laws for the variation of the energy and the linear scale of turbulence downstream of a grid. Laws, based on the Kármán hypotheses, are close, in the intervals where measurements are made, to power laws (they argree with the experimental data at least as well as power laws) and they possess certain advantages from the point of view of the theory of homogeneous turbulence of a viscous incompressible fluid. We show that solutions, based on the Kármán hypotheses, are compatible with the equations for the higher moments, even if we assume for these moments additional hypotheses similar to the Kármán hypotheses.

1. A theory for the decay of a homogeneous isotropic turbulent motion of a viscous incompressible fluid can proceed from various hypotheses relating to the behavior of the velocity correlation moments of the second and third order, for example, the Kármán hypotheses [1]

$$b_d^{\,d}(r, t) = \langle u(0, t)\, u(r, t)\rangle = b(t)\, f(\chi) \qquad (1.1)$$

$$b_d^{\,nn}(r, t) = \langle u(0, t)\, v^2(r, t)\rangle = b^{3/2} h(\chi) \qquad (1.2)$$

$$(b(t) = b_d^{\,d}(0, t) = \langle u^2\rangle = \langle v^2\rangle, \quad \chi = r / l(t))$$

Here r is the distance, t is the time, u is the projection of the velocity pulsation in the r direction and v is that in the direction normal to r. The angle brackets denote an averaging operation.

Based on the hypotheses (1.1) and (1.2), a derivation was given by Sedov [2, 3], without making additional hypotheses, of possible laws of variation for the functions $b(t)$, $l(t)$, $f(\chi)$ and $h(\chi)$ (two cases were involved: in the first, $b \sim (t - t_0)^{-1}$ and $l \sim (t - t_0)^{1/2}$, and f and h were related through an equation with constant coefficients; in the second case, a complete solution of the problem was obtained). In [4] this solution was generalized to the case of nonisotropic turbulence.

A comparison of the theoretical results obtained using (1.1) and (1.2), which was given in [4, 5], shows that this complete solution gives good argreement with the experimental data on the decay of turbulent motions of air and water downstream of grids (for Reynolds numbers not too large). In many of the other papers [6 – 14], on the other hand, the experimental data on the variation of the energy $1.5b$ and the linear scale of turbulence were approximated by the power laws

$$b \sim (t - t_0)^{-n}, \quad l \sim (t - t_1)^m \tag{1.3}$$

not coinciding with the laws of Sedov's complete solution. Moreover, in [15], on the basis of power law handling of the experimental data [10, 11], a conclusion was arrived at directly contradicting the results given in [4, 5] concerning the nonapplicability of the Kármán hypothesis (1.2).

In the present paper we explain this contradiction by showing that the results given in [15] exceed the accuracy limits for the comparison of theory with experiment. We discuss the power laws (1.3), point out their deficiencies from the point of view of modelling homogeneous turbulence, and we point out the proximity of these laws, in the intervals in which measurements were made, to the laws of Sedov's complete solution based on the Kármán hypotheses.

2. Theoretical and experimental investigations in the late forties and early fifties by Batchelor, Townsend and Stewart, which resulted in a series of papers and books, led, in particular, to the following results: (1) in some interval of variation of r the Kármán hypotheses (1.1), (1.2) are satisfied (see [6 – 8]), so that one of the cases of Sedov's solution must be realized; (2) the turbulence decay process downstream of a grid can be divided into three stages: an initial stage in which the first case of this solution is realized with the laws $b \sim (t - t_0)^{-1}$, $l \sim (t - t_0)^{0.5}$ (actually, this is the region in which the curve $b^{-1}(t)$ is close to its tangent (*) ; a transition stage, not described theoretically; and a final stage, in which third order moments can be neglected, with the laws $b \sim (t - t_0)^{-5/2}$, $l \sim (t - t_0)^{1/2}$ (essentially, this is the region where the curve $b^{-0.4}(t)$ is close to its tangent).

A criticism of this type of representation is given in [1, 5], where it was shown that the second case of Sedov's solution (or its generalization to the case of homogeneous nonisotropic turbulence), yielding the same laws $b(t)$ and $l(t)$) with a minimum of assumptions, gives a complete and compact description of the whole turbulence decay process downstream of a grid. However, a criticism of this representation has developed

*) In Batchelor's book [16] the notion of the initial stage is not connected with the hypotheses (1.1), (1.2).

in another direction, namely, from the experimental work. It was shown that under the experimental conditions which Batchelor and Townsend considered sufficient for realization of the final stage the third order moments cannot be neglected, and that the laws for energy variation under these conditions may be approximated better by the power law expressions $b \sim (t - t_0)^{-n}$, where $n \leqslant 2$ (see [9 – 11]). On the other hand, in describing the initial stage the same power law expressions with $n > 1$ have been employed [12 – 14]. In particular, it was shown that a better description of Batchelor and Townsend's experimental data could be had with $1.1 < n < 2$ [9 – 12]. Thus, of the original concepts of final and initial turbulence decay stages only the names remain.

3. The power law expressions, adopted in many of the experimental papers, are, of course, acceptable as good approximations of the experimental data. At the same time, from the point of view of the theory of homogeneous (not necessarily isotropic) turbulence, these expressions show certain deficiencies which are not present in the laws of Sedov's complete solution [2, 3], these laws being based on the Kármán hypotheses (1.1), (1.2) and being close to the power laws (1.3) in the intervals for which measurements were taken. Since the expressions (1.3) do not agree for $t_0 \neq t_1$ or $m \neq {}^1/_2$ with the complete Kármán-Howarth equation, which describes the dynamics of the decay of homogeneous isotropic turbulence of a viscous incompressible fluid (or the analog of this equation for nonisotropic turbulence), we consider in more detail a particular case of the expressions (1.3) ised in treating the experimental data [10, 11]

$$b \sim (t - t_0)^{-n}, \qquad l \sim (t - t_0)^{1/2} \tag{3.1}$$

On the basis of an approximation of the experimental data by the expression (3.1) and the function $f = (1 + \chi^2)^{-1}$, and on the basis also of the expression $b_d^{nn} = (t - t_0)^{-n-0.5} h(\chi)$ [10, 11], which follows from these approximations and from the Karmán-Howarth equation, the authors of [15] proposed adding to the hypothesis (1.1) the hypothesis

$$b_d^{nn}(r, t) = b_1^{3/2}(t)\, h(\chi) \tag{3.2}$$

They concluded that the Kármán hypothesis (1.2) was not acceptable (i.e. one cannot take $b \equiv b_1$ in the relation (3.2)).

Much of the published experimental data was analyzed and used in [4, 5], including that given in [10, 11]. The detailed comparison of the theory with the experimental data concerning $b(t)$, $\lambda(t) \sim l(t)$ and $f(\chi)$ (and, in one case, concerning $h(\chi)$), which was presented in [4, 5], confirms the acceptability of the Kármán hypotheses (1.1) and (1.2); therefore, the fundamental conclusion in [15] concerning the need for replacing the hypothesis (1.2) by the hypothesis (3.2) cannot be regarded as sufficiently reliable. This is connected with the fact that the conclusions made by the authors of [15] exceed the limits of accuracy in the comparison of theory with experiment.

In [4, 5] a comparison was made of Sedov's solution, based on the Kármán hypotheses, only with those experiments in which the quantity f was measured and only for values of the time greater or equal to those for which the hypothesis (1.1) was experimentally confirmed. A determination of the parameters b^* and λ^* according to the scheme employed in [4, 5] (from the comparison of the theoretical curve $\lambda(b)$ with the corresponding experimental points) was effected fairly simply and graphically; however, it does not give any better degree of accuracy: the mean square deviation of the experimental points $b(t)$, $\lambda(t)$ (see [10, 11]) from the theoretical ones amounted to about 10%. If we take into account the experimental errors (around 10%) connected with finiteness of the averaging time, the effect of noise, and the departures from homogeneity and isotropicity, we can regard this accuracy in the comparison of theory and experiment as allowable. Nevertheless, we made yet another comparison of the theory with the experiments of [10, 11], the aim being: (1) to increase the accuracy of the correspondence between theory and experiment; (2) to increase the time interval in which the theoretical laws $b(t)$ and $\lambda(t)$ agree with experiment; (3) to establish a correspondence between the theory and experiments in which the quantity f is not measured. As a universal parameter α for all the experiments of [10, 11] we used $\alpha_- = 0.08$ [4, 5]. The parameters b^*, λ^*, t^* were chosen from the condition that the quantity $\Sigma\,[1 - t(b_k)/t_k]^2$ (b_k and

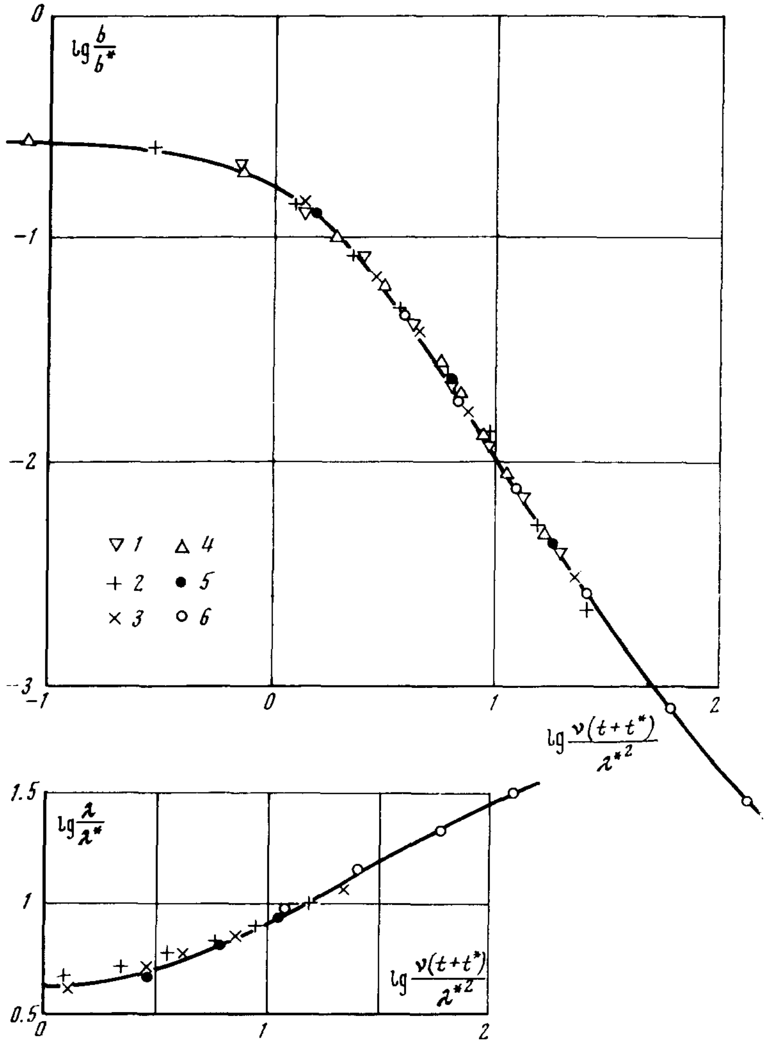

Fig. 1

t_k are experimental points, $t(b_k)$ is the curve $\alpha_- = 0.08$ [4, 5].

As is evident from both the Fig. 1 and the Table 1, wherein the more precise values of the parameters of the theory are used, and where Fig. 1 includes both the experimental points and the mean square deviations ε_b and ε_λ of the experimental points $b(t)$ and $\lambda(t)$ from the theoretical curves, the experimental data of [10, 11] show good agreement with the solution based on the Kármán hypotheses. The deviations of the experimental points $b(t)$, for suitable t^*, from the theoretical curve (for one and the same $\alpha_- = 0.08$) do not exceed the corresponding deviations from the power law relationships (in which the values of n are different: $n = 2, 1.73, 1.35$). In the last row of the table the maximum values of Ut/M are given. Discarding the measured b and λ data in the immediate vicinity of the grid (for $Ut/M < (Ut/M)_{max}$) is based on the generally accepted idea that it is possible to compare the theory of isotropic turbulence with experimental data according to the decay of the turbulence beyond the grid only when $Ut/M > 20$—30.

Table 1

Type of grid	A	B	C	A+B*	$V_p/U = 3$	$V_p/U = 17$
R_m	940	470	840	—	2000	2000
M/d	2.8	2.8	5	—	5	5
U, cm/sec	2.9	2.9	2.9	—	3.14	3.14
M, cm	3.56	1.78	3.18	—	6.4	6.4
t^*, sec	−22.2	−17.9	−11.8	−19.7	−5.9	1.25
U^2/b^*	15.9	20	84.7	29.4	33.3	0.445
λ^{*2}/ν, sec	14.3	10.9	6.97	17.1	8.12	1.57
ε_b, %	4.5	8	1	3.6	1	2
ε_λ, %	—	9	6	—	2,5	3.8
Number of the point	1	2	3	4	5	6
$(Ut/M)_{max}$	17	25	14	12**	4	***

*) The grid A is 30 cm in front of the grid B; time is referred to grid A;
**) The value for grid A is shown;
***) All measurements taken into account.

The following notation is employed in the table: Ut is the distance from the grid; M is the grid spacing; d is the diameter of the grid rods; U is the flow velocity relative to the grid, V_p is a typical speed of movong grids [11]; and $R_M = UM/\nu$.

In addition we note the following:

1°. Power law treatment of the experimental data in [10, 11], and the similar treatment in [15], yields the following low precision relationship for the third order moments:

$$b_1 / b = (t - t_0)^{(n-1)/3}$$

Approximation of the same experimental data in [4, 5] and in the present paper gives a constant value for the ratio b_1/b. There are no direct experimental measurements of b_1. In the experimental data treated in [10, 11] the exponent $(n-1)/3$ is small. The ratio b_1/b (over a time interval in which the hypothesis (1.1) is confirmable from the experimental data) is found to be within the limits of accuracy for comparing theory with experiment from a knowledge of $b(t)$, $l(t)$ and $f(\chi)$, whereas the main conclusion in [15] is based on a difference in the laws for b and b_1 at the expense of a small exponent $(n-1)/3$.

2°. For power laws the case $n=1$ corresponds to $b=b_1$. In Sedov's theory, $b=b_1$ and no power laws in $t+t^*$ are obtained (t^* is determined experimentally; moreover, $t^* \neq -t_0$). It is important to emphasize that in the theory based on the equality $b=b_1$ (with an appropriate t^*) theoretically correct laws are also obtained, with a minimum of assumptions, even for correlation functions. But if the power laws (3.1) are used, then, within the scope of the initial assumptions (1.1) and (3.2) made in [15], correct theoretical expressions for the correlation functions do not exist at all. We remark that the theoretical curve $f(\chi)$, used in [4, 5], and the empirical curve of [10, 11] are practically coincident. The difference does not exceed 0.015, which is less than the error in the measurements associated with finiteness of the averaging time and is equal to $0{,}03\sqrt{1+f^2}$.

3°. The power laws (3.1) for the third order moments lead to the expression

$$b_d^{nn}(r_0, t) = \langle u(0, t)\, v^2(r_0, t)\rangle = b_1^{3/2} h(\chi_0) \sim (t-t_0)^{-n-0.5} \qquad (3.3)$$

for an arbitrary constant value $\chi_0 = r_0(t)/l(t) \neq 0$. Since in the experiments no decrease in the region of applicability of the hypothesis (1.1) was observed relative to χ with an increase in t, it therefore follows that we can regard the quantity χ_0 in the expression (3.3) as being constant. Using the strict inequality

$$[b_d^{nn}(r, t)]^2 \leqslant \langle u^2(0,t)\rangle \langle v^4(0,t)\rangle$$

we find from the relation (3.3) that $\langle v^4\rangle / \langle v^2\rangle^2 \to \infty$ as $t \to \infty$ no slower than $(t-t_0)^{n-1}$. This result contradicts theoretical notions concerning a Gaussian distribution for pulsations of the velocity v and all the available data concerning the fact that $\langle v^4\rangle / \langle v^2\rangle^2 \approx 3$ with an accuracy of around 10% (see [10] and also [17], p. 223).

4. A theory of homogeneous isotropic turbulence can proceed only from the Kármán-Howarth equation without involving the equations for the higher moments of the correlation of the pulsations of the velocity and the pressure. Nevertheless, there can arise the question of satisfying the equations for the higher order moments. One can assume the Kármán hypotheses for the second and third moments without necessarily assuming these same hypotheses for the higher moments (as is also the case for the hypotheses concerning homogeneity and isotropicity) [2, 3]. All the analogs, in fact, of the Kármán hypotheses can be applied, after the Kármán-Howarth equation, to the equations in the chain of equations for the higher moments.

By way of example, we consider the equation of the dynamics of two-point third order moments

$$\frac{\partial b_i^{mn}}{\partial t} - 2\nu \frac{\partial^2 b_i^{mn}}{\partial r_j\, \partial r_j} + \nu T_i^{mn} + S_i^{mn} = 0 \qquad (4.1)$$

$$b_i^{mn} = \langle u_i u_m' u_n'\rangle, \qquad T_i^{mn} = \left\langle u_i \frac{\partial u_m'}{\partial r_j} \frac{\partial u_n'}{\partial r_j}\right\rangle$$

$$S_i^{mn} = \frac{\partial}{\partial x_j}\langle u_i u_j u_m' u_n'\rangle + \frac{\partial}{\partial x_j'}\langle u_i u_j' u_m' u_n'\rangle + \frac{\partial}{\partial x_i}\langle P u_m' u_n'\rangle +$$

$$\left\langle u_i u_n' \frac{\partial P'}{\partial x_m'} + u_i u_m' \frac{\partial P'}{\partial x_n'}\right\rangle \qquad (r_k = x_k' - x_k;\quad i, j, m, n, k = 1, 2, 3)$$

Here u_i, P and u_i', P' are the velocity and pressure pulsations at the points with the Cartesian coordinates x_k and x_k', respectively, at the time t. The Kármán hypothesis (1.2) gives

$$b_i^{mn}(r_k, t) = b^{3/2}(t)\, f_i^{mn}(\chi_k), \qquad \chi_k = r_k / l(t)$$

Similarly, we suppose that

$$T_i^{mn}(r_k, t) = b^{3/2} l^{-2} \varphi_i^{mn}(\chi_k), \quad S_i^{mn}(r_k, t) = b^2 l^{-1} \psi_i^{mn}(\chi_k) \tag{4.2}$$

Using the system of equations for $b(t)$ and $l(t)$ [2, 3], which follows from the Kármán hypotheses (1.1) and (1.2), we find, for the first case of the solution (when $b \sim (t - t_0)^{-1}$), an equation with constant coefficients, binding the tensors f_i^{mn}, φ_i^{mn} and ψ_i^{mn}, and, for the second case of the (complete) solution, we find

$$\varphi_i^{mn} = 15\alpha f_i^{mn} + \frac{\chi_k}{2} \frac{\partial f_i^{mn}}{\partial \chi_k} + 2 \frac{\partial^2 f_i^{mn}}{\partial \chi_k\, \partial \chi_k}$$

$$\psi_i^{mn} = -p\chi_k \frac{\partial f_i^{mn}}{\partial \chi_k}, \quad f_i^{mn} = \left(h - \chi \frac{\partial h}{\partial \chi}\right) \frac{\chi_i \chi_m \chi_n}{\chi^3} +$$

$$\frac{1}{2\chi} \frac{\partial (\chi^2 h)}{\partial \chi} \left(\delta_{im} \frac{\chi_n}{\chi} + \delta_{in} \frac{\chi_m}{\chi}\right) - h\delta_{mn} \frac{\chi_i}{\chi}$$

$$h = \frac{2\alpha p \chi}{4\alpha - 1} \left\{ {}_1F_1\left[10\alpha; \frac{5}{2}; -\frac{\chi^2}{8}\right] - {}_1F_1\left[10\alpha + 1; \frac{7}{2}; -\frac{\chi^2}{8}\right] \right\},$$

$$\chi^2 = \chi_k \chi_k$$

Here α and p are the parameters of Sedov's solution [2 – 5], and ${}_1F_1$ is the degenerate hypergeometric function [18].

Thus, with the aid of the hypotheses (4.2) (which are analogous to the Kármán hypotheses) we can, in the case of the complete (for the second and third moments) solution, based on the Kármán hypotheses, find the tensors T_i^{mn} and S_i^{mn}. We note that, following this, the fourth moments of the velocity correlation (for example $\langle u_i u_j u_m' u_n' \rangle$) are determined with complete arbitrariness. We can arrive at a similar result if, instead of Eq. (4.1) for the two-point tensor $\langle u_i u_m' u_n' \rangle$, we use the equation (see, for example, [17]) for the three-point tensor $\langle u_i u_m' u_n'' \rangle$ with corresponding similarity hypotheses.

The author thanks L. I. Sedov for a useful discussion on the results of this paper.

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Translated by J. F. H.